8.2. Determine the Galois groups of the following polynomials over $\mathbb{Q}$:
(a) $x^3 - 2$, **(b)** $x^3 + 3x + 14$, **(c)** $x^3 - 3x^2 + 1$, **(d)** $x^3 - 21x + 7$,
(e) $x^3 + x^2 - 2x - 1$, **(f)** $x^3 + x^2 - 2x + 1$.

8.3. Determine the quadratic polynomial $q(x)$ that appears in (16.8.2) explicitly, in terms of α_1 and the coefficients of f.

8.4. Let $K = \mathbb{Q}(\alpha)$, where α is a root of the polynomial x^3+2x+1, and let $g(x) = x^3+x+1$. Does $g(x)$ have a root in K?

8.5. Let α_i be the roots of a cubic polynomial $f(x) = x^3 + px + q$. Find a formula for a second root α_2 in terms of the elements α_1, δ, and the coefficients of f.

Section 9 Quartic Equations

9.1. Let K be a Galois extension of F whose Galois group is the symmetric group S_4. Which integers occur as degrees of elements of K over F?

9.2. With reference to Example 16.9.2**(a)**, write the element $\alpha + \alpha'$ as a nested square root. What other nested square roots does K contain?

9.3. Can $\sqrt{4+\sqrt{7}}$ be written in the form $\sqrt{a} + \sqrt{b}$, with rational numbers a and b?

9.4. **(a)** Prove that the polynomial $x^4 - 8x^2 + 11$ is irreducible over $\mathbb{Q}$ in two ways: using the methods of Chapter 12 and computing with its roots.
(b) Do the same for the polynomial $x^4 - 8x^2 + 9$.
(c) Determine all intermediate fields when K is the splitting field of $x^4 - 8x^2 + 11$ over $\mathbb{Q}$.

9.5. Consider a nested square root $\alpha = \sqrt{r+\sqrt{t}}$ with r and t in a field F. Assume that α has degree 4 over F, let f be the irreducible polynomial of α over F, and let K be a splitting field of f over F.

(a) Compute the irreducible polynomial $f(x)$ for α over F. Prove that $G(K/F)$ is one of the groups D_4, C_4, or D_2.
(b) Explain how to determine the Galois group in terms of the element $r^2 - t$.
(c) Assume that the Galois group of K/F is the dihedral group D_4. Determine generators for all intermediate fields $F \subset L \subset K$.

9.6. Compute the discriminant of the quartic polynomial $x^4 + 1$, and determine its Galois group over $\mathbb{Q}$.

9.7. Assume that an extension field K/F has the form $K = F(\sqrt{a}, \sqrt{b})$. Determine all nested square roots $\sqrt{r+\sqrt{t}}$ that are in K, with r and t in F.

9.8. Determine whether or not the following nested radicals can be written in terms of unnested square roots, and if so, find an expression.
(a) $\sqrt{2+\sqrt{11}}$, **(b)** $\sqrt{10+5\sqrt{2}}$, **(c)** $\sqrt{11+6\sqrt{2}}$, **(d)** $\sqrt{6+\sqrt{11}}$, **(e)** $\sqrt{11+\sqrt{6}}$.

* **9.9.** **(a)** Determine the discriminant and the resolvent cubic of a polynomial of the form $f(x) = x^4 + rx + s$.
(b) Determine the Galois groups of $x^4 + 8x + 12$ and $x^4 + 8x - 12$ over $\mathbb{Q}$.
(c) Can the roots of the polynomial $x^4 + x - 5$ be constructed by ruler and compass?